# The Thue-Morse Sequence

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## Class Discussion

A magic trick, where the magician guesses the fifth card with the help of an assistant.

## The Thue-Morse Sequence

This is a summer homework. It is made in a style of theme rounds.

**Exercise 1.** On Monday the baby said A, on Tuesday AU, on Wednesday AUUA, on Thursday AUUAUAAU. What will she say on Saturday?

You can see that this very gifted baby increases her talking capacity twice each day. If the baby continues indefinitely, her text converges to an infinite sequence that mathematicians call the Thue-Morse sequence. Of course, mathematicians use zeroes and ones instead of A and U, so the sequence looks like 0110100110010110100. . ..

**Exercise 2.** Prove that if you replace every zero by 01 and every one by 10 in the Thue-Morse sequence, you will get the Thue-Morse sequence back.

**Exercise 3.** Prove that every other term reproduces the whole sequence.

Why do you think this sequence is called a fractal sequence?

**Exercise 4.** The Thue-Morse sequence is a fixed point under the substitution in the previous exercise. Prove that the only two fixed points under this substitution are the Thue-Morse sequence and its negation. (By the way, what do you think a negation might mean?)

**Exercise 5.** Prove that the sequence doesn't contain substrings 000 and 111.

**Exercise 6.** Prove that the sequence doesn't contain any cubes. That is, it doesn't contain XXX, where X is any binary string.

**Exercise 7.** Prove that the sequence doesn't contain any overlapping squares, that is strings 1X1X1 and 0X0X0, where X is any binary string.

**Exercise 8.** Prove that if the Thue-Morse sequence contains a string X, it contains an infinite number of copies of X.

**Exercise 9.** Prove that if the Thue-Morse sequence contains a string X, it contains an infinite number of negations of X.

A number is evil if the number of ones in its binary expansion is even, and odious if it's odd. We can define a function, called the odiousness of a number, in the following way: odiousness($n$) is one, if $n$ is odious and 0 otherwise.

**Exercise 10.** Prove that the Thue-Morse sequence is the odiousness of the sequence of non-negative numbers.

**Exercise 11.** Let's say that the Thue-Morse sequence is related to binary numbers and as such corresponds to $k = 2$. Can you describe an analogue of the sequence for $k = 3$ or $k = 4$? If so, can you find and prove the properties similar to above for these cases?

**Exercise 12.** Consider a sequence $a(n)$. We can build from it another sequence: $b(n) = \text{odiousness}(a(n))$. For example, if $a(n) = 2^n$, then $b(n) = 1$. Find other sequence $a(n)$, such that $b(n)$ can be easily described.