

Test

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May 23, 2016

Your name:

Your grade:

Warm-up

Exercise 1. 1 point. Everything I said or will say before 2017 will become a lie on Jan 1, 2017. Was this statement true or a lie?

Answer:

Exercise 2. 1 point. I draw several triangles on a piece of paper. First I showed the paper to Lev and asked him how many triangles are there. Lev said 5 and he was right. Then I showed the paper to Sasha and asked him how many triangles are there. Sasha said 3 and he was right. How many triangles are there on the paper?

Answer:

Exercise 3. 1 point. It takes me 2 minutes to get up the stairs from the first floor to the third floor. Assuming that I am very fit and will not change my climbing speed, how much time does it take me to get from the first floor to the 8th floor?

Answer:

Exercise 4. 1 point. I was driving with the speed of 60 miles an hour. What should be my speed if I want to pass each mile one minute faster?

Answer:

Exercise 5. 1 point. John and Jim are twins. How many grandparents do they have?

Answer:

Exercise 6. 2 points. In the game of Inky Pinky, the first player offers a concise, clear definition and the second player must translate that definition into two words that rhyme. The first player also indicates the number of syllables in each word by saying “Ink Pink” for one-syllable words, “Inky Pinky” for two-syllable words and so on. For example: Question: a yearly handbook. Inkity Pinkity. Answer: an annual manual.

Solve the following questions:

- Inexpensive land vehicle. Ink Pink.
- A dumb little boy with a bow and arrow. Inky Pinky.
- Frozen bike. Inkity Pinkity.
- Royal cloth. Inkity Pinkity.

Answer:

Cryptography and Number Theory

Exercise 7. 1 point. Decipher: Guvfvfgurpbeerpgnafjre.

Answer:

Exercise 8. 1 point. Prove that numbers $27x + 4$ and $18x + 3$ are coprime for any integer x .

Answer:

Exercise 9. 1 point. What can you say if the number of divisors of a number including 1 and itself is 2? 3?

Answer:

Exercise 10. 2 points. Find remainders:

1. 19^{14} modulo 7
2. 19^{10} modulo 66

Answer:

Exercise 11. 1 point. To get to your account at a bank in Wonderland you need to dial a 7-digit password. The bank disconnects the phone as soon as you dial a wrong digit. What is your strategy to get to an account? In how many tries are you guaranteed to brake the password?

Answer:

Exercise 12. 2 points. How many positive integers less than 10,000 are neither squares nor cubes?

Answer:

Algebra

Exercise 13. 2 points. Find the remainder of the polynomial $P(x) = x^{81} + x^{27} + x^9 + x^3 + x$ when dividing by a) $x - 1$, b) $x^2 - 1$.

Answer:

Exercise 14. 1 point. Find a and b such that the equation $x^3 + ax + b = 0$ has three distinct roots that form an arithmetic progression.

Answer:

Fair Division

Exercise 15. 1 point. Alice and Bob are divorcing. They want to divide everything fairly. They own a house. Bob values the house at its market price of \$400,000. Alice loves the neighborhood and hates moving, so she values the house at \$440,000. How much money would Alice owe Bob if she gets the house?

Answer:

Geometry

Exercise 16. 1 point. One of the legs of a right triangle is 10 meters longer than the other leg, and 10 meters shorter than the hypotenuse. Find the hypotenuse.

Answer:

Exercise 17. 1 point. Prove that at least one number in a Pythagorean triple has to be divisible by 5.

Answer:

The Thue-Morse sequence

Exercise 18. 1 point. On Monday the baby said A, on Tuesday AU, on Wednesday AUUA, on Thursday AUUAUAAU. What will she say on Saturday?

Answer:

You can see that this very gifted baby increases her talking capacity twice each day. If the baby continues indefinitely, her text converges to an infinite sequence that mathematicians call the Thue-Morse sequence. Of course, mathematicians use zeros and ones instead of A and U, so the sequence looks like 0110100110010110100...

Exercise 19. 2 points. Prove that every other term reproduces the whole sequence.

Solution:

Exercise 20. 2 points. Prove that the sequence doesn't contain substrings 000 and 111.

Solution: