Fermat's Little Theorem

Tanya Khovanova

November 9, 2015

Nothing produces such odd results as trying to get even.

Class Discussion

Theorem (Fermat's Little Theorem)**.** *If p is a prime number, then for any integer a the number $a^p - a$ is divisible by p.*

Warm Up

Exercise 1. Mrs. Fullhouse has 2 sons, 3 daughters, 2 cats and 1 dog. How many children does she have?

Exercise 2. Take 9 from 6, 10 from 9, 50 from 40 and leave 6.

Exercise 3. What is the value of 1/2 of 2/3 of 3/4 of 4/5 of 5/6 of 6/7 of 7/8 of 8/9 of 9/10 of 1000?

Exercise 4. 2001 AMC 8. Two dice are thrown. What is the probability that the product of the two numbers is a multiple of 5?

Exercise 5. 2002 AMC 8. What is the measure of the acute angle formed by the hands of a clock at 4:20 a.m.?

Number Theory

Exercise 6. Use Fermat's Little Theorem to find:

- the remainder of 19^{52}, when divided by 53,
- the remainder of 3^{63}, when divided by 61,

- the remainder of 4^{119}, when divided by 59.

Exercise 7. Use Fermat's Little Theorem and Chinese Remainder theorem to find:

- the remainder of 13^{180}, when divided by 209,
- the remainder of 3^{73}, when divided by 91.

Exercise 8. Prove that for any integer a, $a^5 - a$ is divisible by 30 and $a^{11} - a$ is divisible by 66.

Exercise 9. Given a sequence $a_n = 1 + 2^n + 3^n + 4^n + 5^n$, is it possible to find five consecutive terms of this sequence such that all of them are divisible by 2010?

Exercise 10. Prove that for any integer n that is not a multiple of 17, that either $n^8 + 1$ or $n^8 - 1$ is divisible by 17.

Exercise 11. Let p be a prime number and a is not a multiple of p. Prove that there exists an integer b such that $ab \equiv 1 \pmod{p}$.

Exercise 12. Number a is three times its sum of digits. Prove that a is divisible by 27.

Competition Practice

Exercise 13. 2002 HMMT. Find the greatest common divisor of the numbers $2002 + 2$, $2002^2 + 2$, $2002^3 + 2$, $\ldots$.

Exercise 14. 1999 HMMT. A combination lock has a 3-number combination, with each number an integer between 0 and 39 inclusive. Call the numbers n_1, n_2, and n_3. If you know that n_1 and n_3 leave the same remainder when divided by 4, and n_2 and $n_1 + 2$ leave the same remainder when divided by 4, how many possible combinations are there?

Exercise 15. 2007 HMMT. Define the sequence of positive integers a_n recursively by $a_1 = 7$ and $a_n = 7^{a_{n-1}}$ for all $n \geq 2$. Determine the last two digits of a_{2007}.

Exercise 16. 2002 HMMT. For how many integers a $(1 \leq a \leq 200)$ is the number a^a a square?