Invariants III

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Invariants

Exercise 1. There are six numbers on the blackboard: 1, 2, 3, 4, 5, 6. In one move you are allowed to add 1 to any two of them. How many moves do you need to make all numbers equal each other?

Exercise 2. A knight starts on the chessboard at the $a1$ cell. Can it tour all the cells of the chessboard exactly once and end at the $h8$ cell?

Exercise 3. Numbers are put into an m by n table so that the sum in every row is 1, as well as the sum in every column is 1. Prove that $m = n$.

Exercise 4. Alice and Bob are playing a game. The game starts with two piles of pebbles: with 10 pebbles and 15 pebbles. Each turn a person is allowed to divide one pile into two smaller piles. Alice starts, and the person who doesn't have a move loses. Can Bob win?

Exercise 5. I've put 100 coins in one row. In one move you are allowed to switch two coins that are separated by exactly one coin in between them. How can you put all the coins in the reverse order?

Exercise 6. Every door in a house leads from a room to a room, or from a room to outside. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

Exercise 7. Can you cut a 10 by 10 board into 1 by 4 rectangles?

Exercise 8. Start with the positive integers $1, \ldots, 4n - 1$. In one move you may replace any two integers by their difference. Prove that after $4n - 2$ steps, the final remaining integer will be even.

Exercise 9. At first, a dance hall is empty. Each minute, either a person comes in or two people start dancing. After exactly 100 minutes, could the room have exactly 50 non-dancing people in the hall?

Exercise 10. Numbers 1 and 2 are written on the board in the office of the math professor Bob Enstein. Every day Bob's graduate student has to replace two numbers on the board with their arithmetic and harmonic means. If a and b are numbers, then $(a+b)/2$ is their arithmetic mean, and $2/(1/a+1/b)$ is their harmonic mean. Once Bob came to his office and noticed that one of the numbers got erased. What is the erased number if the other number is 941664/665857? Is it possible that Bob will ever see 35/24 as one of the numbers on the board?

Exercise 11. An ace of spades is on the table face up. You are allowed to roll it over the edge several times. Can it end up in the initial spot face down? Can it end up in the initial spot face up, but with the picture upside down?

Exercise 12. Can you cover a 10 by 10 board with one corner cut out by 1 by 3 rectangles? What about an 8 by 8 board with one corner cut out?

Competition Practice

Exercise 13. 2010 Tournament at braingames.ru. Only truth-tellers and liars live on a certain island. The truth-tellers always tell the truth; the liars always lie. Every person on the island lives in a four-story building. The island government conducted a census and every inhabitant participated. The were four yes/no questions in the census. These are the percentages of yeses.

- Do you live on the first floor? — 40%.
- Do you live on the second floor? — 30%.
- Do you live on the third floor? — 50%.
- Do you live on the fourth floor? — 0%.

What percentage of population lives on the first floor?

Exercise 14. 1988 Tournament of Towns. Every vertex of a cube is assigned a number $+1$ or -1. Every face has a number that is the product of all the numbers in its corners. Then the 14 numbers are summed up (all the vertices and faces). Can the sum be 0?