Invariants

Tanya Khovanova

April 27, 2015

Class Discussion

The variation of 12 coins problem when you need to produce the weighings in advance. Mnemonic: MA DO — LIKE, ME TO — FIND, FAKE — COIN.

Remainders mod 9. Other invariants. My 'guess the number' trick. My newspaper trick. Problem together: Start with 7^{2010}. At each step, delete the leading digit, and add it to the remaining number. First, repeat until a number with exactly 10 digits remains. Prove that this number has two equal digits. Second, repeat until you get a single digit. What is it?

- The sum of digits of n has the same remainder modulo 9 as n.
- Ten-digit numbers with all distinct digits are divisible by 9.
- Rearranging digits doesn't change the remainder modulo 9.
- Removing the leading digit and adding it to the rest of the the number doesn't change the remainder mod 9.

Invariants

Exercise 1. Let $SOD(n)$ be the sum of the digits of n. Suppose $f(n)$ is the result of iterating SOD many times until we get a single digit. That is, $f(n) = SOD(SOD(SOD \ldots (n) \ldots))$. Find $f(2^{2010})$, $f(3^{2010})$, $f(4^{2010})$, $f(5^{2010})$, $f(6^{2010})$, $f(7^{2010})$, $f(8^{2010})$, $f(9^{2010})$, and $f(1234^{2009})$.

Exercise 2. A frog jumps along the line. First it jumped 1 cm, then 3 cm in the same or the opposite direction, then 5 cm. It continues with the sequence of odd numbers. Can it be back at the beginning after 57 jumps?

Exercise 3. Prove that the sum of the digits of a square can't be 1967.

Exercise 4. A number has three ones. All other digits are zeroes. Can it be a square?

Exercise 5. The integers from 1 to 2009 are written on a blackboard. You are allowed to erase any two numbers a and b replacing them with $a \times b$. At the end there is one number left on the board. What can it be?

Exercise 6. Can you have 25 korunas in 10 bills of 1, 3 or 5 korunas? Can you have 50 dinars in 10 bills of 1, 4 or 10 dinars?

Exercise 7. Can you put numbers into a rectangular grid in such a way that the sum in every column is negative and in every row is positive?

Exercise 8. All AMSA students have candy in their pockets. Every student has twice more pieces of candy in his/her right pocket than in the left pocket. Can all AMSA students have exactly 1000 pieces of candy together?

Exercise 9. You have a chocolate bar that consists of small squares arranged in a rectangle. You need to split the bar into small squares by each time splitting a piece along the lines between the squares. Given that the rectangle is $m \times n$, what is the smallest number of splits that you will need?

Exercise 10. Is it possible for two different powers of 2 to have the same digits (in a different order)?

Competition Practice

Exercise 11. Prove that for any natural number n, $4^n + 15n - 1$ is divisible by 9.

Exercise 12. 1967 USSR Olympiad. Number b was produced by permuting digits in number a. Can $a + b$ equal $999 \ldots 999$, a number written with 1967 nines? In a similar setting, prove that if $a + b = 10^{10}$ then a is divisible by 10.

Exercise 13. 1962 IMO. Determine the smallest possible integer x whose last decimal digit is 6, and if we erase this last 6 and put it in front of the remaining digits, we get four times x.