Induction

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Class Discussion

Mathematical Induction (base case, inductive step). Different starting steps. Invent a wrong statement that can be "proven" by using a mathematical induction if one forgets to use the base step.

Warm Up

Exercise 1. A stick has two ends. If you cut off one end, how many ends will the stick have left?

Exercise 2. A hypotenuse of a right triangle is 10 inches, and the altitude having the hypotenuse as its base is 6 inches. Find the area of the triangle.

Exercise 3. What is the smallest prime divisor of $5^{2009} + 1$?

Induction

Exercise 4. Use the mathematical induction to prove that the Fibonacci sequence F_n satisfies $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Exercise 5. Use the mathematical induction to prove that the Fibonacci sequence F_n satisfies $F_{n+1}^2 = F_n F_{n+2} + (-1)^n$.

Exercise 6. For integers a and b, prove that if $a^2 + b^2$ is divisible by 3, then it is divisible by 9.

Exercise 7. Use the mathematical induction to prove that $1 + 3 + 5 + ... + (2n - 1) = n^2$.

Exercise 8. How many zeroes at the end does 100! have?

Exercise 9. Use the mathematical induction to prove that $1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n + 1)! - 1$.

Exercise 10. USSR bank has an unlimited number of 3-ruble and 5-ruble bills. Prove that it can pay any number of rubles starting from 8 exactly.

Challenge Problems

Exercise 11. USAMO 2003. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

Exercise 12. Find the flaw in the following induction "proof" that all horses are of the same color.

The case with just one horse is trivial. If there is only one horse in the group, then clearly all horses in that group have the same color.

Inductive step. Assume that k horses always are the same color. Let us consider a group consisting of $k + 1$ horses.

First, exclude the last horse and look only at the first k horses; all these are the same color since k horses always are the same color. Likewise, exclude the first horse and look only at the last k horses. These too, must also be of the same color. Therefore, the first horse in the group is of the same color as the horses in the middle, who in turn are of the same color as the last horse. Hence the first horse, middle horses, and last horse are all of the same color, and we have proven that:

If k horses have the same color, then $k + 1$ horses will also have the same color. Thus in any group of horses, all horses must be the same color.