Test Solutions

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1 Results

29 participated. The easiest exercise was 5: 28 people solved it. The most difficult exercise was 10: 2.5 people solved it. The largest possible score was 24. The largest achieved score 22, the smallest: 1. The average score is 8.8. The scores:

- 22: Aditya Hoque
- 20: Joshua Michel
- 18: Andrew Machkasov
- 15: Ben Maron
- 13: Anjana Shenoy
- 9-12: nine people
- 5-7: eight people
- 2.5-4: six people
- 1: one person

2 Rewards

I already rewarded my students who made it to the AIME: Satwik Mekala, Aditya Hoque, Pranav Nagalamadaka, Kevin Cai, Anjana Shenoy, Zephyr Lucas, Andrew Machkasov.

In addition to that, the following students are rewarded:

- James Rose for the most gold stars.
- Joshua Michel for the second best score on the test.

- Ben Maron for an exceptional performance by my first year student.

Also, I would like to make an exception and reward Aditya again for an exceptional test performance.

3 Solutions to Exercises

Exercise 1. 1 point. A family photo contained: one grandfather, one grandmother, two fathers, two mothers, six children, four grandchildren, two brothers, two sisters, three sons, three daughters, one father-in-law, one mother-in-law, one daughter-in-law.

29 people you may think, but no! What is the fewest number of people that could have been in the photo?

Solution: 8. Four children (2 boys and 2 girls), their mother and father, the mother's mother, and the father's father.

Exercise 2. 1 point. Half of zero is still zero. What other number can be halved to make zero?

Solution: 8. Slice the number 8 horizontally in half. I gave the full credit for -0.

Exercise 3. 1 point. A ship is docked in the harbor. Over the side hangs a rope ladder with rungs a foot apart. The tide rises at a rate of 9 inches per hour. At the end of six hours, how much of the rope ladder will still remain above water, assuming that 9 feet were above the water when the tide began to rise?

Solution: 9. Still 9 feet because the ladder will rise with the ship.

Exercise 4. 1 point. How can you make the following equation correct without changing it at all? $8 + 8 = 91$.

Solution: Look at it upside down: $16 = 8 + 8$.

Exercise 5. 1 point. At noon, you look at the clock in your bedroom. The big hand is on the five and the little hand is in between the 3 and the 4. What time is it?

Solution: Noon. (However, if you answered that it's time to get a new clock, you're right, too.)

Exercise 6. 1 point. There are 2 hourglasses measuring 7 and 4 minutes respectively. How do you measure 5 minutes? Explain.

Solution: You can use the fact that $5 = 4 \cdot 3 - 7$.

Exercise 7. 1 point. How many numbers between 1 and 1000 are not divisible by 3 or 7?

Solution: In this range, there are 333 numbers divisible by 3, 142 numbers divisible by 7, 47 numbers divisible by 21. In the range from 1 to 1000 inclusive, the number of numbers not divisible by 3 or 7 is $1000-333-142+47 = 572$. I forgot to say "inclusive", so the full credit will be given for either 572 or 570.

Exercise 8. 1 point. How many 5-digit numbers are there with at least one odd digit?

Solution: There are $9 \cdot 10^4 = 90000$ 5-digit numbers. There are $4 \cdot 5^4 = 2500$ 5-digit numbers with all even digits. Thus the answer is $90000 - 2500 = 87500$.

Exercise 9. 2 point. A faulty car odometer proceeds from digit 4 to digit 6, always skipping the digit 5, regardless of position. For example, after traveling one mile the odometer changed from 000049 to 000060. If the odometer now reads 002917, how many miles has the car actually traveled?

Solution: The odometer essentially counts in base 9, except digits over 5 should be adjusted. We need to translate 2816 from base 9 to base 10. The answer is 2121.

Exercise 10. 2 points. Count the number of subsets of $\{1, 2, \ldots, 10\}$ that contain no consecutive integers. Explain why.

Solution: Denote A_n the number of subsets of $\{1, 2, \ldots, n\}$ that contain no consecutive integers. Out of those, A_{n-1} subsets do not contain n and A_{n-2} subsets contain n. Thus $A_n = A_{n-1} + A_{n-2}$. Additionally, $A_1 = 2$ and $A_2 = 3$. Thus $A_n = F_{n+2}$: shifted Fibonacci numbers. The answer is $A_{10} = F_{12} = 144$.

Exercise 11. 2 points. The 100 game: two players start from 0 and alternatively add a number from 1 to 10 to the sum. The player who reaches 100 wins. List all P-positions.

Solution: 1, 12, 23, 34, 45, 56, 67, 78, 89.

Exercise 12. 2 points. There are two people, A and B, each whom is either a knight or a knave. A makes the following statement: "At least one of us is a knave." What are A and B?

Solution: A can't be a knave, as in this case this statement is true. Therefore, A is a knight. A's statement is true, therefore B is a knave.

Exercise 13. 2 points. Is number $21^{10} - 1$ divisible by 2200? Explain.

Solution: $21^{10} - 1 = 441^5 - 1 = (441 - 1)(441^4 + 441^3 + 441^2 + 441^1 + 1)$. The first factor is 440, and the last is divisible by 5 as each summand ends in 1.

Exercise 14. 2 points. Tanya decided to buy balloons for her math party. There are 4 colors of balloons at the Star Market and Tanya needs 6 balloons. In how many ways can Tanya buy her balloons?

Solution: This is the bagel problem. The answer is $\binom{9}{3} = 84$.

Exercise 15. 4 points. A group of five friends decide to exchange gifts as secret Santas. Each person writes their name on a piece of paper and puts it in a hat and then each person randomly draws a name from the hat to determine who has them as their secret Santa.

What is the probability that at least one person draws their own name?

Solution: We had this problem as a homework with 5 people, but no one solved it. The first person choses not his/her name with probability 3/4. After that it becomes complicated. It depends on whether the first person chose the gift meant for the second person. On the other hand, we studied derangements: the number of ways for everyone to not get their own names. If the total number of people is four, then the names could cycle (there are 6 ways to do that), or there could be two pairs of people that swap their names (there are 3 ways to do that). The total number of derangements is 9. The answer is $(24 - 9)/24 = 5/8$.