Invariants 2

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Class Discussion

The variation of 12 coins problem when you need to produce the weighings in advance. Mnemonic: MA DO — LIKE, ME TO — FIND, FAKE — COIN.

Warm Up

Exercise 1. The integers from 1 to 2009 are written on a blackboard. You are allowed to erase any two numbers a and b replacing them with $a \times b$. At the end there is one number left on the board. What can it be?

Exercise 2. Can you have 25 korunas in 10 bills of 1, 3 or 5 korunas? Can you have 50 dinars in 10 bills of 1, 4 or 10 dinars?

Exercise 3. Can you put numbers into a rectangular grid in such a way that the sum in every column is negative and in every row is positive?

Exercise 4. All AMSA students have candy in their pockets. Every student has twice more pieces of candy in his/her right pocket than in the left pocket. Can all AMSA students have exactly 1000 pieces of candy together?

Exercise 5. You have a chocolate bar that consists of small squares arranged in a rectangle. You need to split the bar into small squares by each time splitting a piece along the lines between the squares. Given that the rectangle is $m \times n$, what is the smallest number of splits that you will need?

Competition Practice

Exercise 6. 2010 Tournament at braingames.ru. Only truth-tellers and liars live on a certain island. The truth-tellers always tell the truth; the liars always lie. Every person on the island lives in a four-story building. The island government conducted a census and every inhabitant participated. The were four yes/no questions in the census. These are the percentages of yeses.

- Do you live on the first floor? — 40%.
- Do you live on the second floor? — 30%.
- Do you live on the third floor? — 50%.
- Do you live on the fourth floor? — 0%.

What percentage of population lives on the first floor?

Exercise 7. Start with 7^{2010}. At each step, delete the leading digit, and add it to the remaining number.

- Repeat until a number with exactly 10 digits remains. Prove that this number has two equal digits.
- Repeat until you get a single digit. What is it?

Exercise 8. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

Challenge Problems

Exercise 9. Can you cut a 10 by 10 board into 1 by 4 rectangles?

Exercise 10. Is it possible for two different powers of 2 to have the same digits (in a different order)?

Exercise 11. Start with the positive integers 1, $\ldots$, $4n - 1$. In one move you may replace any two integers by their difference. Prove that after $4n-2$ steps, the final remaining integer will be even.

Exercise 12. At first, a dance hall is empty. Each minute, either a person comes in or two people start dancing. After exactly 100 minutes, could the room have exactly 50 non-dancing people in the hall?